MAXIMAL QUADRATIC MODULES ON ∗-RINGS

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ABSTRACT. We generalize the notion of and results on maximal proper quadratic modules from commutative unital rings to ∗-rings and discuss the relation of this generalization to recent developments in noncommutative real algebraic geometry. The simplest example of a maximal proper quadratic module is the cone of all positive semidefinite complex matrices of a fixed dimension. We show that the support of a maximal proper quadratic module is the symmetric part of a prime ∗-ideal, that every maximal proper quadratic module in a Noetherian ∗-ring comes from a maximal proper quadratic module in a simple artinian ring with involution and that maximal proper quadratic modules satisfy an intersection theorem. As an application we obtain the following extension of Schmüdgen's Strict Positivstellensatz for the Weyl algebra: Let c be an element of the Weyl algebra $\mathcal{W}(d)$ which is not negative semidefinite in the Schrödinger representation. It is shown that under some conditions there exists an integer k and elements $r_1, \ldots, r_k \in \mathcal{W}(d)$ such that $\sum_{j=1}^{k} r_j c r_j^*$ is a finite sum of hermitian squares. This result is not a proper generalization however because we don't have the bound $k \leq d$.

1. INTRODUCTION

The aim of this note is to generalize the notion of and results on quadratic modules from commutative unital rings to associative unital rings with involution, which we call ∗-rings. The study of quadratic modules in ∗-rings is suggested by the recent developments in noncommutative real algebraic geometry, see [1], [8], [5], [16], [17].

Commutative real algebraic geometry is based on the notion of an ordering and quadratic modules are considered just a technical tool. However, an attempt by M. Marshall to build a noncommutative real algebraic geometry on ∗-orderings in [13] showed that there is not enough

of them. The advantage of maximal proper quadratic modules over $*$-orderings is not only in their quantity but also in their connection to the representation theory of $*$-rings, see [2].

The exposition can be divided into three parts. In the first part (Sections 2 and 3) we define a quadratic module in a $*$-ring and provide elementary examples. We also try to generalize the following result (cf. [19, 1.1.4 Theorem]): If M is a maximal proper quadratic module in a commutative ring R with trivial involution, then $M \cup -M = R$ and $M \cap -M$ is a prime ideal. It will be shown that the first property cannot be generalized but the second can.

In the second part (Sections 4 and 5) we show that every maximal quadratic module in a Noetherian $*$-ring comes from a maximal quadratic module in a simple artinian ring with involution using a variant of Goldie's theory from [6]. In the commutative case, this is rather obvious. Namely, by factoring out the prime ideal $M \cap -M$ we get a maximal proper quadratic module in $R/M \cap -M$ and by passing to the field F of fraction of $R/M \cap -M$ we get a maximal proper quadratic module in F, both times in a natural way.

In the third part (Sections 6 and 7) we generalize the following intersection theorem (see [9, 1.8 Satz]): An element r of a commutative ring R with trivial involution belongs to $R \setminus -M$ for every maximal proper quadratic module M in R if and only if there exist elements $m, t \in R$ which are sums of squares of elements from R and satisfy $tr = 1 + m$. As an application, we obtain the following extension of Schmüdgen's Strict Positivstellensatz for the Weyl algebra (see [16, Theorem 1.1]):

Theorem. *Let $\mathcal{W}(d)$ be the d-th Weyl algebra with the natural involution and let π_0 be its Schrödinger representation. If c is a symmetric element of $\mathcal{W}(d)$ of even degree $2m$ then the following are equivalent:*

(1) *$\pi_0(c)$ is not negative semidefinite and the highest degree part $c_{2m}(z, \overline{z})$ of c is strictly positive for all $z \in \mathbb{C}^d$, $z \neq 0$.*
(2) *There exist elements $r_1, \ldots, r_k, s_0, s_1, \ldots, s_l \in \mathcal{W}(d)$ such that $\sum_{j=1}^{k} r_j c r_j^* = \sum_{i=0}^{l} s_i s_i^*$ and $\pi_0(s_0)$ is invertible and $\deg(s_0) \geq \deg(s_j)$ for every $j = 1, \ldots, l$.*

An early version of this manuscript was presented in Luminy and Saskatoon in March 2005. Meanwhile, our techniques have also been applied to the free $*$-algebra, see [10].

2. Definitions and elementary examples

Let R be a $*$-ring and $\mathrm{Sym}(R) := \{r \in R \mid r = r^*\}$ its set of symmetric elements. When no confusion is possible we write S for $\mathrm{Sym}(R)$. A

subset M of R is a *quadratic module* if $M \subseteq S$, $1 \in M$, $M + M \subseteq M$ and $rMr^* \subseteq M$ for every $r \in R$. (This is very similar to the definition of an *m-admissible wedge* in [15, page 22].) A quadratic module M is *proper* if $-1 \notin M$. The smallest quadratic module in R is $N(R) := \{\sum_i r_i r_i^* : r_i \in R\}$. The quadratic module $N(R)$ need not be proper. If $N(R)$ is proper, then R is *semireal*. A proper quadratic module is *maximal* if it is not contained in any strictly larger proper quadratic module.

Remark. The following properties of R are equivalent:

   (1) R is semireal,
   (2) R has at least one proper quadratic module,
   (3) R has at least one maximal proper quadratic module.

Example. Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded operators on H with the standard involution. Then the set $P(H)$ of all positive definite operators from H is clearly a proper quadratic module in $B(H)$.

If H is finite dimensional, then $P(H)$ is maximal. Namely, if A is a symmetric operator which is not in $P(H)$ then we can find a basis of H such that the matrix of A is diagonal with first entry equal to -1. Let E_{ij} be the operator whose matrix in this basis has (i,j)-th entry equal to 1 and all other entries equal to zero. Note that $\sum_k E_{k1} A E_{k1}^* = -I$ where I is the identity operator on H. Hence, every quadratic module that contains A also contains $-I$. Therefore, there is no proper quadratic module stricly larger then $P(H)$. Note that in the finite dimensional case $P(H) = N(B(H))$, hence $P(H)$ is the only proper quadratic module in $B(H)$.

If H is infinite dimensional, then $P(H)$ is not maximal. Namely if K_s is the set of all symmetric compact operators on H, then $M = P(H) + K_s$ is a proper quadratic module in $B(H)$ that is strictly larger than $P(H)$. If $-1 \in M$ then there exists $A \in P(H)$ such that $-I - A$ is compact. This is not possible because the eigenvalues of a compact operator tend to zero while the eigenvalues of $-I - A$ are bounded away from zero.

Remark. If R is a commutative unital ring with trivial involution (i.e. $\mathrm{Sym}(R) = R$) and M is a maximal proper quadratic module in R, then $M \cup -M = \mathrm{Sym}(R)$. This property fails in the noncommutative case. Namely, if H is a finite dimensional Hilbert space of dimension at least two then $P(H)$ is a maximal proper quadratic module on $B(H)$ such that $P(H) \cup -P(H) \neq \mathrm{Sym}(B(H))$.

3. The support

Let M be a proper quadratic module in a $*$-ring R. The set

$$M^0 = M \cap -M$$

is called *the support* of M. We will frequently use the following properties of M^0: $M^0 + M^0 \subseteq M^0$, $-M^0 \subseteq M^0$, $rM^0r^* \subseteq M^0$ for every $r \in R$, if $x + y \in M^0$ and $x, y \in M$ then $x \in M^0$ and $y \in M^0$. Write

$$J_M = \{a \in R \mid auu^*a^* \in M^0 \text{ for all } u \in R\}.$$

Proposition 1. *If M is a proper quadratic module in R, then J_M is a two-sided ideal in R containing the set M^0.*

Proof. If $a, b \in J_M$ then for every $u \in R$, $auu^*a^* \in M^0$ and $buu^*b^* \in M^0$. Write $x = (a + b)uu^*(a + b)^*$ and $y = (a - b)uu^*(a - b)^*$. Since $x + y = 2(auu^*a^* + buu^*b^*) \in M^0$ and $x, y \in N(R) \subseteq M$, it follows that $x \in M^0$ and $y \in M^0$ for any $u \in R$. Hence $a + b \in J_M$ and $a - b \in J_M$.

If $a \in J_M$ and $r \in R$ then for every $u \in R$, $(ra)uu^*(ra)^* = r(auu^*a^*)r^* \in M^0$ and $(ar)uu^*(ar)^* = a(ru)(ru)^*a^* \in M^0$, so that $ra \in J_M$ and $ar \in J_M$. Hence, J_M is a two-sided ideal.

To prove that $M^0 \subseteq J_M$, we must show that $auu^*a \in M^0$ for every $a \in M^0$ and $u \in R$. Pick $a \in M^0$ and $u \in R$ and write $z = uu^*$, $x = (1 + az)a(1 + az)^*$ and $y = (1 - az)a(1 - az)^*$. Note that $x, y \in M^0$ and $4aza = x - y \in M^0$. Since $aza \in M$, it follows that $aza \in M^0$. □

Recall that a two-sided ideal J of a ring R is *prime* if for any $a, b \in R$ such that $aRb \subseteq J$ we have that either $a \in R$ or $b \in R$. A ring R is *prime* if (0) is a prime ideal of R.

Theorem 1. *If M is a maximal proper quadratic module in R, then the ideal J_M is prime and $*$-invariant. Moreover, $J_M \cap S = M^0$.*

Proof. We already know that J_M is a two-sided ideal containing M^0. Hence, $M^0 \subseteq J_M \cap S$. If $J_M \cap S \not\subseteq M^0$, then there exists $s \in J_M \cap S$ such that $s \notin M^0$. Replacing s by $-s$ if necessary, we may assume that $-s \notin M$. Since M is maximal, it follows that the smallest quadratic module containing M and $-s$ is not proper. Hence there exist an element $n \in M$, an integer l and elements $t_1, \ldots, t_l \in R$ such that $1 + n = \sum_{j=1}^{l} t_j s t_j^*$. Since J_M is a two-sided ideal containing s, it follows that $1 + n \in J_M$. It follows that $(1 + n)vv^*(1 + n) \in M^0$ for every $v \in R$. For $v = 1$, we get $(1 + n)^2 \in M^0$. Since $n, n^2 \in M$, it follows that $1 \in -M$, contradicting the assumption that M is proper. Therefore, $J_M \cap S = M^0$.

To prove that J_M is prime, pick $a, b \in R$ such that $arb \in J_M$ for every $r \in R$. If $b \notin J_M$, then there exists $v \in R$ such that $bvv^*b^* \notin M^0$. Since

M is maximal and $-bvv^*b^* \notin M$, it follows that the smallest quadratic module containing M and $-bvv^*b^*$ is not proper. Hence, there exist an element $m \in M$, an integer k and elements $r_1, \ldots, r_k \in R$ such that $1 + m = \sum_{i=1}^{k} r_i(bvv^*b^*)r_i^*$. Pick $r \in R$ and write $x = arr^*a^*$ and $y = armr^*a^*$. Since $arr_ib \in J_M$ for every $i = 1, \ldots, k$ by assumption and $J_M \cap S = M^0$ by the first paragraph of this proof, it follows that $x + y = ar(1 + m)r^*a^* = \sum_{i=1}^{k}(arr_ib)vv^*(arr_ib)^* \in M^0$. Clearly, $x, y \in M$, so that $x, y \in M^0$. Therefore, arr^*a^* for every $r \in R$, implying that $a \in J_M$.

To show that J_M is ∗-invariant, pick $a \in J_M$. Since J_M is an ideal, it follows that $a^*uu^*a \in J_M$ for every $u \in R$. By the first paragraph of this proof, $J_M \cap S = M^0$, so that $a^*uu^*a \in M^0$ for every $u \in R$. So, $a^* \in J_M$ by the definition of J_M. $\qquad\square$

Remark. If R is a complex ∗-algebra and M is a maximal proper quadratic module in R, then $J_M = M^0 + iM^0$. Namely, pick $z \in J_M$ and write $z = x + iy$ where $x, y \in S$. Then $z^* = x - iy$ also belongs to J_M. Pick any $r \in R$ and write $s = rr^*$. Since $zsz^* \in M^0$ and $z^*sz \in M^0$, it follows that $2(xsx + ysy) = zsz^* + z^*sz \in M^0$. Since $xsx, ysy \in M$, it follows that $xsx, ysy \in M^0$. Hence $x, y \in J_M$. The relation $J_M \cap S = M^0$ implies that $x, y \in M^0$. The converse is clear.

4. Quadratic modules and rings of fractions

We assume that the reader is familiar with the definition of a reversible Ore set and Ore localization, see Section 1.3 of [3]. The aim of this section is to discuss consequences of the following observation:

Proposition 2. *Let M be a proper quadratic module in a ∗-ring A and let N be a ∗-invariant reversible Ore set on A such that $N \cap M^0 = \emptyset$. Let $Q = AN^{-1}$ and $\tilde{M} = \{q \in \mathrm{Sym}(Q)|\ nqn^* \in M \text{ for some } n \in N\}$. Then $\tilde{M}$ is a proper quadratic module in Q.*

Proof. Clearly, the involution extends uniquely from A to Q, see also [6]. To prove that $\tilde{M} + \tilde{M} \subseteq \tilde{M}$ take $q_1, q_2 \in \tilde{M}$. Pick $n_1, n_2 \in N$ such that $n_1q_1n_1^* \in M$ and $n_2q_1n_2^* \in M$. By the Ore property of N, there exist $u \in A$ and $v \in N$ such that $un_1 = vn_2$. It follows that $vn_2(q_1 + q_2)(vn_2)^* = u(n_1q_1n_1^*)u^* + v(n_2q_1n_2^*)v^* \in M$. Since $vn_2 \in N$, we have that $q_1 + q_2 \in \tilde{M}$.

Suppose that $q \in \tilde{M}$ and $d = n^{-1}a \in Q$. Pick $z \in N$ such that $zqz^* \in M$. By the Ore property of N, there exist $b \in A$ and $w \in N$ such that $bz = wa$. Then, $(wn)(dqd^*)(wn)^* = waq(wa)^* = b(zqz^*)b^* \in M$ and $wn \in N$. Hence, $dqd^* \in \tilde{M}$.

If $-1 \in \tilde{M}$, then there exists $n \in N$ such that $-nn^* \in M$. It follows that $nn^* \in M^0 \cap N$ contrary to the assumption that $M^0 \cap N = \emptyset$. $\square$

Let A, N and Q be as in Proposition 2. Then for every proper quadratic module M' in Q we have

$$\left(M' \cap \mathrm{Sym}(A) \right)^{\tilde{}} = M'.$$

On the other hand, for every quadratic module M in A such that $M^0 \cap N = \emptyset$, the set

$$\overline{M} := \tilde{M} \cap \mathrm{Sym}(A) = \{a \in \mathrm{Sym}(A) | \ nan^* \in M \text{ for some } n \in N\}$$

is also a quadratic module in A such that $\overline{M}^0 \cap N = \emptyset$, which we call the N-*closure* of M. We say that M is N-*closed* if $M^0 \cap N = \emptyset$ and $M = \overline{M}$. Note that N-closure is an idempotent operation.

Theorem 2. *Let A, N, Q be as above. The mappings $M \mapsto \tilde{M}$ and $M' \mapsto M' \cap \mathrm{Sym}(A)$ give a bijective correspondence between proper N-closed quadratic modules of A and proper quadratic modules in Q.*

5. A REPRESENTATION THEOREM

Suppose that R is a prime Noetherian $*$-ring and N the set of all elements from R that are not zero divisors. A variant of Goldie's Theorem from [6] says that N is a $*$-invariant reversible Ore set, the involution of A extends uniquely to the Ore's localization $Q = RN^{-1}$ and there exists a skew–field D such that Q is either isomorphic to $M_n(D)$ or $*$-isomorphic to $M_n(D) \oplus M_n(D)^{op}$ with exchange involution $(a, b)^* = (b, a)$. If R is *real*, i.e. it has a support zero quadratic module, then by Proposition 2 this quadratic module extends to a proper quadratic module of Q. Note that $M_n(D) \oplus M_n(D)^{op}$ with involution $(a, b)^* = (b, a)$ cannot have a proper quadratic module because of the identity $(-1, -1) = (1, -1)(1, -1)^*$. Hence:

Proposition 3. *The Goldie ring of fractions of a real prime Noetherian $*$-ring is isomorphic to $M_n(D)$ for an integer n and a skew-field D.*

The main result of this section is the following representation-theoretic characterization of maximal proper quadratic modules in Noetherian $*$-rings:

Theorem 3. *Let A be a Noetherian $*$-ring and M a proper quadratic module in A. The following are equivalent:*

   (1) *M is maximal,*

(2) *there exists a simple artinian ring with involution $Q \cong M_n(D)$, a maximal proper quadratic module M' in Q and a $*$-ring homomorphism $\pi \colon A \to Q$ such that*

$$M = \pi^{-1}(M') \cap \operatorname{Sym}(A).$$

Proof. $(1) \Rightarrow (2)$: If M is maximal then by Theorem 1 J_M is a prime $*$-ideal such that $J_M \cap \operatorname{Sym}(A) = M^0$. Let $j \colon A \to A/J_M$ be the canonical projection. Then $j(M)$ is a maximal support zero quadratic module on A/J_M and $M = j^{-1}(j(M)) \cap \operatorname{Sym}(A)$. Let N be the set of all non-zero divisors in A/J_M and $Q = (A/J_M)N^{-1}$. By Proposition 2 and Theorem 2, $M' = j(M)\tilde{\ }$ a maximal proper quadratic module in Q, $j(M) = M' \cap j(A)$ and Q is a simple artinian ring with involution. Let $i \colon A/J_M \to Q$ be the canonical imbedding. Then $i^{-1}(M') = M' \cap j(A) = j(M)$. Write $\pi = i \circ j$ and note that $\pi \colon A \to Q$ is a $*$-ring homomorphism such that $M = \pi^{-1}(M') \cap \operatorname{Sym}(A)$.

$(2) \Rightarrow (1)$: Follows from Theorem 2. $\qquad\square$

Theorem 3 reduces the study of maximal proper quadratic modules in Noetherian $*$-rings into two parts: the study of their supports and the study of maximal proper quadratic modules in simple artinian rings with involution. In the special case of PI $*$-rings, the simple artinian rings in part two are finite dimensional (i.e. central simple algebras). Quadratic modules on central simple algebras with involution will be studied in a separate paper.

6. Intersection theorem

Intersection theorems for quadratic modules in commutative rings with trivial involution have been considered in [9]. Theorem 4 generalizes [9, 1.8 Satz].

Theorem 4. *Let R be a semireal ring and S its set of symmetric elements. For every $x \in S$ and every proper quadratic module M_0 the following are equivalent:*

(1) $x \in S \setminus -M$ *for every maximal proper quadratic module M containing M_0,*
(2) $-1 \in M_0 - N[x]$, *where $M_0 - N[x]$ is the smallest quadratic module containing M_0 and $-x$.*

Proof. If (2) is false, then $-1 \notin M_0 - N[x]$, hence $M_0 - N[x]$ is a proper quadratic module. By Zorn's Lemma, there exists a maximal proper quadratic module M containing $M_0 - N[x]$. The fact that $x \in -M$ implies that (1) is false. Conversely, if (1) is false, then $x \in -M$ for

some maximal proper quadratic module M. It follows that $M_0 - N[x] \subseteq M$, so that $-1 \notin M_0 - N[x]$. Hence, (2) is false.   □

Intersection theorems are popularly called Stellensätze. M. Schweighofer proposed the name *abstract Nirgendsnegativsemidefinitheitsstellensatz* for our Theorem 4.

In the archimedean case we can reformulate Theorem 4 by using the representation theorem from [2]. Recall that a proper quadratic module M is *archimedean* if for every element $a \in R$ there exists an integer n such that $n - aa^* \in M$. In Section 2 of [2], it is shown that for every archimedean quadratic module M in a complex $*$-algebra R, there exists an M-positive irreducible $*$-representation ϕ_M of R on a complex Hilbert space. Recall that a representation ϕ is *M-positive* if $\phi(a)$ is positive semidefinite for every $a \in M$.

Theorem 5. *For every archimedean proper quadratic module M_0 in a complex $*$-algebra R and for every $x \in \mathrm{Sym}(R)$, the following are equivalent:*

  (1) *For every M_0-positive irreducible representation ψ of R, $\psi(x)$ is not negative semidefinite.*
  (2) *There exists $k \in \mathbb{N}$ and $r_1, \ldots, r_k \in R$ such that $\sum_{i=1}^{k} r_i x r_i^* \in 1 + M_0$.*

Proof. If (1) is false, then there exists an M_0-positive irreducible $*$-representation ϕ such that $\phi(x)$ is negative semidefinite. Hence, ϕ is $M_0 - N[x]$ positive. If follows that $-1 \notin M_0 - N[x]$, so that (2) is false.

Conversely, if (2) is false then $-1 \notin M_0 - N[x]$, so that $M_0 - N[x]$ is a proper quadratic module containing M_0. By Zorn's Lemma, there exists a maximal proper quadratic module M containing $M_0 - N[x]$. Clearly, M is archimedean. By the discussion above, there exists an M-positive irreducible $*$-representation ϕ of R. Clearly, ϕ is M_0-positive and $\phi(-x)$ is positive semidefinite. Hence, (1) is false.   □

7. An application of the intersection theorem

The aim of this section is to prove a variant of Schmüdgen's Strict Positivstellensatz for the Weyl algebra, see [16, Theorem 1.1]. We will refer to Schmüdgen's original proof several times.

Recall that the d-th Weyl algebra $\mathcal{W}(d)$ is the unital complex $*$-algebra with generators $a_1, \ldots, a_d, a_{-1}, \ldots, a_{-d}$, defining relations $a_k a_{-k} - a_{-k} a_k = 1$ for $k = 1, \ldots, d$ and $a_k a_l = a_l a_k$ for $k, l = 1, \ldots, d$, $-1, \ldots, -d$, $k \neq -l$ and involution $a_k^* = a_{-k}$ for $k = 1, \ldots, d$. Write deg for the total degree in the generators. The graded algebra that corresponds to the filtration by deg is $\mathbb{C}[z, \bar{z}] = \mathbb{C}[z_1, \ldots, z_d, \bar{z}_1, \ldots, \bar{z}_d]$, a

polynomial algebra in $2d$ complex variables. Write $N = a_1^* a_1 + \ldots + a_d^* a_d$ and fix $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Let $\mathcal{N}$ be the set of all finite products of elements $N + (\alpha + n)1$, where $n \in \mathbb{Z}$. Let Φ be the Fock-Bargmann representation of $\mathcal{W}(d)$. It is unitarily equivalent to the Schrödinger representation π_0. Let $\mathcal{X}$ be the unital complex ∗-algebra generated by $y_n = \Phi(N + (\alpha + n)1)^{-1}$ for $n \in \mathbb{Z}$ and $x_{kl} = \Phi(a_k a_l)y_0$, $k, l = 1, \ldots, d, -1, \ldots, -d$. By [16, Lemma 3.1], $N(\mathcal{X})$ is an archimedean quadratic module in $\mathcal{X}$.

Theorem 6. *Let c be a symmetric element of $\mathcal{W}(d)$ with even degree $2m$ and let c_{2m} be the polynomial from $\mathbb{C}[z, \bar{z}]$ that corresponds to the $2m$-th component of c. The following assertions are equivalent:*

(1) *$\pi_0(c)$ is not negative semidefinite and $c_{2m}(z, \bar{z}) > 0$ for all $z \in \mathbb{C}^d$, $z \neq 0$.*

(2) *There exist $k, l \in \mathbb{N}$ and elements $r_1, \ldots, r_k, s_0, \ldots, s_l \in \mathcal{W}(d)$ such that $\sum_{i=1}^k r_i^* c r_i = \sum_{j=0}^l s_j^* s_j$ and $s_0 \in \mathcal{N}$ and $\deg(s_0) \geq \deg(s_j)$ for every $j = 1, \ldots, l$.*

Proof. $(1) \Rightarrow (2)$: If c has degree $4n$ then by [16, Lemma 3.2] the element $\tilde{c} := y_0^n \Phi(c) y_0^n$ belongs to $\mathcal{X}$. The main part of the proof is to show that there exist elements $h_i \in \mathcal{X}$ such that $\sum_i h_i \tilde{c} h_i^* \in 1 + N(\mathcal{X})$. (Then we get (2) by clearing out the denominators using the identities $\Phi(a_j) y_k = y_{k+1} \Phi(a_j)$ and $\Phi(a_j)^* y_k = y_{k-1} \Phi(a_j)^*$.) If this claim is false, then by Theorem 5, there exists a representation π of $\mathcal{X}$ such that $\pi(\tilde{c})$ is negative semidefinite. By [16, Section 4], we know that π can be decomposed as $\pi_1 \oplus \pi_\infty$ where π_1 is a sum of "identity" representations and π_∞ is an integral representation defined by $\pi_\infty(\tilde{c}) = \int_{S^d} c(z, \bar{z}) dE(z, \bar{z})$ where E is a spectral measure on the sphere S^d. By [16, Section 5], we have that $\pi_\infty(\tilde{c}) = \pi_\infty(c_{4n})$ where c_{4n} is the leading term of c. Since $c_{4n}(z, \bar{z}) > 0$ for every $z \in S^d$ by the second assumption, there exists by the compactness of S^d an $\epsilon > 0$ such that $c_{4n} \geq \epsilon$. It follows that $\langle \pi_\infty(\tilde{c})\phi, \phi \rangle \geq \int_{S^d} \epsilon d\langle E(z, \bar{z})\phi, \phi \rangle = \epsilon \int_{S^d} d\|E(z, \bar{z})\phi\|^2 = \epsilon \|\phi\|^2$ for every $\phi \in L^2(\mathbb{R}^d)$. Since $\pi(\tilde{c})$ is negative semidefinite and $\pi_\infty(\tilde{c})$ is positive definite, it follows that $\pi_1(\tilde{c})$ is nontrivial and negative definite. Since π_1 is a direct sum of identity representations, it follows that $\tilde{c} < 0$. Since y_0^n has dense image, it follows that $\Phi(c) \leq 0$. Hence $\pi_0(c) \leq 0$ by the unitary equivalence of Φ and π_0, a contradiction with assumption (1). If c has degree $4n + 2$, then we replace c by $\sum_{j=1}^d a_j c a_j^*$ and proceed as above.

$(2) \Rightarrow (1)$: Since $s_0 \in \mathcal{N}$, $\pi_0(s_0)$ is invertible. It follows that $\pi_0(\sum_{j=0}^l s_j^* s_j) > 0$. Since $\sum_{i=1}^k r_i^* c r_i = \sum_{j=0}^l s_j^* s_j$ it follows that also $\pi_0(\sum_{i=1}^k r_i^* c r_i) > 0$. Hence $c \not\leq 0$ as claimed. Write $t = \deg(s_0)$

and note that $(s_0^* s_0)_{2t}(z, \bar{z}) = (\sum_{n=1}^{d} z_n \bar{z}_n)^t > 0$ for $z \neq 0$, Since $t \geq \deg(s_j)$ for every $j = 1, \ldots, l$, it follows that $(\sum_{j=0}^{l} s_j^* s_j)_{2t}(z, \bar{z}) > 0$ for $z \neq 0$. From $\sum_{i=1}^{k} r_i^* c r_i = \sum_{j=0}^{l} s_j^* s_j$ and $(\sum_{i=1}^{k} r_i^* c r_i)_{2t}(z, \overline{z}) = (\sum_{i=1}^{k} r_i^* r_i)_{2t-2m}(z, \overline{z}) c_{2m}(z, \overline{z})$ we get that $c_{2m}(z, \overline{z}) > 0$ for $z \neq 0$. $\square$

Remark. In [16, Theorem 1.1] both (1) and (2) are stronger. In (1) the assumption $\pi_0(c) \not\leq 0$ is replaced by $\pi_0(c - \epsilon \cdot 1) > 0$ for some ϵ and in (2) the bound $k \leq d$ is provided. The implication from (2) to (1) is not considered.

We believe that the main result of [17] can be extended in a similar way. It would also be interesting to know whether the second part of assertion (1) ($c_{2m} > 0$) implies the first part ($\pi_0(c) \not\leq 0$).

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